TITLE:

Geometry of Banach spaces and norms of $\pm$ matrices

AUTHOR(S):

TAKAHASHI, Yasuji; KATO, Mikio

CITATION:

TAKAHASHI, Yasuji ...[et al]. Geometry of Banach spaces and norms of $\pm$ matrices. 数理解析研究所講究録 1998, 1039: 165-169

ISSUE DATE:

1998-04

URL:

http://hdl.handle.net/2433/61996

RIGHT:

京都大学学術情報リポジトリ KURENAI 紅

Kyoto University Research Information Repository

数理解析研究所講究録
1039 巻 1998 年 165-169

Geometry of Banach spaces and norms of ±1 matrices

Yasuji TAKAHASHI (髙橋泰嗣) and Mikio KATO (加藤幹雄)

Okayama Prefectural University

Kyushu Institute of Technology

Some geometrical properties of a Banach space X are described by the behavior of norms of certain ±1 matrices between $l_r^n(X)$-spaces.

1. Definitions and notations

X, Y ; Banach spaces, $1 \leqq r, s \leqq \infty$, $1/r + 1/r' = 1/s + 1/s' = 1$,

$A = (a_{ij})$; m×n matrix, $a_{ij} = \pm 1$ (m×n ±1 matrix)

$\|A\|_{r,s;X} = \|A : l_r^n(X) \to l_s^m(X)\|$, $l_r^n(X)$; X-valued l_r^n-space

Banach-Matur distance : For two isomorphic Banach spaces E and F

$d(E,F) = \inf\{\|T\| \cdot \|T^{-1}\|$; T is isomorphism from E onto F$\}$.

Finite-representability : Y is finitely representable (f. r.) in X if for each (some) $\lambda > 1$ and for each finite-dimensional subspace F of Y there is a finite-dimensional subspace E of X with dim E = dim F such that $d(E,F) < \lambda$.

Super-reflexivity : X is super-reflexive if any Banach space f. r. in X is reflexive.

Remark. Consider (P), a property of Banach spaces. (P) is said to be a super-property if X has (P) and Y is f. r. in X, then Y has (P). B-convexity and J-convexity are super-properties, but reflexivity and

Radon-Nikodym property (RNP) are not super-properties.

A finite sequence of signs ε_1, ε_2, ..., ε_n is admissible if all + signs are before all - sings (so there are n different sequences).

(1) X ; (n, ε)-convex, $\varepsilon > 0 \Leftrightarrow \forall x_1, \ldots, x_n$ with $\|x_j\| = 1$

$$\min_{\varepsilon_j=\pm 1} \|\Sigma_{j=1}^{n} \varepsilon_j x_j\| \leqq n(1 - \varepsilon).$$

(2) X ; B_n-convex $\Leftrightarrow$ X ; (n, ε)-convex for some $\varepsilon > 0$.

(3) X ; B-convex $\Leftrightarrow$ X ; B_n-convex for some n.

(4) X ; J_n-convex $\Leftrightarrow$ $\exists\, \delta > 0$, $\forall x_1, x_2, \ldots, x_n$ with $\|x_j\| = 1$,

$\exists\, \varepsilon_1, \varepsilon_2, \ldots\ldots, \varepsilon_n$; admissible signs

$$\|\Sigma_{j=1}^{n} \varepsilon_j x_j\| \leqq n(1 - \delta).$$

(5) X ; J-convex $\Leftrightarrow$ X ; J_n-convex for some n.

(6) X ; uniformly non-square $\Leftrightarrow$ X ; B_2-convex ($\Leftrightarrow$ J_2-convex).

Remark. It is known that X is B-convex if and only if l_1 is not f. r. in X; and X is J-convex if and only if it is super-reflexive. It is also known that

uniformly convex $\Rightarrow$ uniformly non-square $\Rightarrow$ super-reflexive

and conversely, any super-reflexive Banach space admits an equivalent uniformly convex norm (see [1] and [2]).

2. Main results

Theorem 1 Let X be a Banach space, $1 < r \leqq \infty$ and $1 \leqq s < \infty$.

Then for any $m \times n$ ± 1 matrix $A = (a_{ij})$, the following are equivalent.

(1) $\exists \delta > 0$, $\forall x_1, \ldots\ldots, x_n$ with $\|x_j\| = 1$

$$\min_{1 \leqq i \leqq m} \| \Sigma_{j=1}^{n} a_{ij} x_j \| \leqq n(1 - \delta).$$

(2) $\|A\|_{r,s;X} = \|A : l_r^n(X) \to l_s^m(X)\| < m^{1/s} n^{1/r'}$.

Remark. For any Banach space X and for any $m \times n$ ± 1 matrix $A = (a_{ij})$, we can prove the following:

(1) $\|A\|_{r,s;X} \leqq m^{1/s} n^{1/r'}$.

(2) If l_1 is f. r. in X, then $\|A\|_{r,s;X} = m^{1/s} n^{1/r'}$.

Corollary 2 (1) X is B_n-convex if and only if $\|A\|_{r,s;X} < m^{1/s} n^{1/r'}$ for some $m \times n$ ± 1 matrix $A = (a_{ij})$.

(2) Let R_n be a $2^n \times n$ Rademacher matrix and $1 < p < \infty$. Then X is B_n-convex if and only if $\|R_n\|_{p,p;X} < 2^{n/p} n^{1/p'}$.

Remark. Rademacher matrices R_n is defined by

$$R_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad R_{n+1} = \left(\begin{array}{c|c} 1 & \\ \vdots & R_n \\ 1 & \\ \hline -1 & \\ \vdots & R_n \\ -1 & \end{array}\right) \quad (n=1, 2, \ldots\ldots).$$

For the details of Rademacher matrices, see Kato, Miyazaki and Takahashi [5].

Corollary 3 (Smith and Turett [9]) Let $1<p<\infty$. Then X is B_n-convex if and only if $L_p(X)$ is. In particular, X is uniformly non-square if and only if $L_p(X)$ is, and X is B-convex if and only if $L_p(X)$ is.

Let A_2 be a 2×2 Littlewood matrix, that is, $A_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Then $n\times n$ admissible matrices A_n ($n\geqq 2$) is defined by

$$A_{n+1} = \left(\begin{array}{c|ccc} 1 & & & \\ 1 & & A_n & \\ \vdots & & & \\ \hline 1 & -1 & -1 \cdots & -1 \end{array} \right) \qquad (n=1, 2, \ldots.)$$

Corollary 4 X is J_n-convex if and only if $\|A_n\|_{p,p;X} < n$ for each (some) p with $1 < p < \infty$. In particular, X is super-reflexive if and only if $\|A_n\|_{p,p;X} < n$ for some n and each (some) p with $1 < p < \infty$.

Corollary 5 (Pisier [8]; see also Beauzamy [1]) Let $1<p<\infty$. Then X is J_n-convex if and only if $L_p(X)$ is. In particular, X is super-reflexive if and only if $L_p(X)$ is.

Theorem 6 Let X be a Banach space and $1 < r \leqq p \leqq s < \infty$. Then for any Bochner space $L_p(X)$ and for any $m\times n$ ± 1 matrix $A = (a_{ij})$, it holds $\|A : l_r^n(L_p(X)) \to l_s^m(L_p(X))\| \leqq \|A : l_r^n(X) \to l_s^m(X)\|$.

Remark. (1) Corollaries 3 and 5 also follow from Theorem 6.

(2) From Theorem 5, it follows that if (p, p')-Clarkson inequality holds in X, then it also holds in $L_r(X)$, where $1 \leqq p \leqq 2$ and $p \leqq r \leqq p'$ (see Takahashi and Kato [10]).

References

[1] B. Beazamy, Introduction to Banach spaces and their geometry, 2nd Ed., North Holland, 1985.

[2] P. Enflo, Banach spaces which can be given an equivalent uniformly convex norm, Israel J. Math. 13 (1972), 281-288.

[3] R. C. James, Uniformly non-square Banach spaces, Ann. of Math. 80 (1964), 542-550.

[4] R. C. James, Super-reflexive Banach spaces, Canad. J. Math. 24 (1972), 896-904.

[5] M. Kato, K. Miyazaki and Y. Takahashi, Type, Cotype Constants for $L_p(L_q)$, Norms of the Rademacher Matrices and Interpolation, Nihonkai Math. J. 6 (1995), 81-95.

[6] M. Kato and Y. Takahashi, On the von Neumann-Jordan constant for Banach spaces, Proc. Amer. Math. Soc. 125 (1997), 1055-1062.

[7] M. Kato and Y. Takahashi, Von Neumann-Jordan constant for Lebesgue-Bochner spaces, to appear in J. Inequalities and Appl.

[8] G. Pisier, Martingales with values in uniformly convex spaces, Israel J. Math. 20 (1975), 326-350.

[9] M. A. Smith and B. Turett, Rotundity in Lebesgue-Bochner function spaces, Trans. Amer. Math. Soc. 257 (1980), 105-118.

[10] Y. Takahashi and M. Kato, Clarkson and random Clarkson inequalities for $L_r(X)$, to appear in Math. Nachr.